

# A comparison of deformations and geometric study of associative algebras varieties

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*The aim of this paper is to give an overview and to compare the different deformation theories of algebraic structures. We describe in each case the corresponding notions of degeneration and rigidity. We illustrate these notions with examples and give some general properties. The last part of this work shows how these notions help in the study of associative algebras varieties. The first and popular deformation approach was introduced by M. Gerstenhaber for rings and algebras using formal power series. A noncommutative version was given by Pinczon and generalized by F. Nadaud. A more general approach called global deformation follows from a general theory of Schlessinger and was developed by A. Fialowski in order to deform infinite dimensional nilpotent Lie algebras. In a Nonstandard framework, M. Goze introduced the notion of perturbation for studying the rigidity of finite dimensional complex Lie algebras. All these approaches has in common the fact that we make an "extension" of the field. These theories may be applied to any multilinear structure. We shall be concerned in this paper with the category of associative algebras.*

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## 1 Introduction

Throughout this paper  $\mathbb{K}$  will be an algebraically closed field, and  $\mathcal{A}$  denotes an associative  $\mathbb{K}$ -algebra. Most examples will be in the finite dimensional case, let  $V$  be the underlying  $n$ -dimensional vector space of  $\mathcal{A}$  over  $\mathbb{K}$  and  $(e_1, \dots, e_n)$  be a basis of  $V$ . The bilinear map  $\mu$  denotes the multiplication of  $\mathcal{A}$  on  $V$ , and  $e_1$  is the unity. By linearity this can be done by specifying the  $n^3$  structure constants  $C_{ij}^k \in \mathbb{K}$  where  $\mu(e_i, e_j) = \sum_{k=1}^n C_{ij}^k e_k$ . The associativity condition limits the sets of structure constants,  $C_{ij}^k$  to a subvariety of  $\mathbb{K}^{n^3}$  which we denote by  $Alg_n$ . It is generated by the polynomial relations

$$\sum_{l=1}^n C_{ij}^l C_{lk}^s - C_{il}^s C_{jk}^l = 0 \text{ and } C_{1i}^j = C_{i1}^j = \delta_{ij} \quad 1 \leq i, j, k, s \leq n.$$

This variety is quadratic, non regular and in general nonreduced. The natural action of the group  $GL(n, \mathbb{K})$  corresponds to the change of basis : two algebras  $\mu_1$  and  $\mu_2$  over  $V$  are isomorphic if there exists  $f$  in  $GL(n, \mathbb{K})$  such that :

$$\forall X, Y \in V \quad \mu_2(X, Y) = (f \cdot \mu_1)(X, Y) = f^{-1}(\mu_1(f(X), f(Y)))$$

The orbit of an algebra  $\mathcal{A}$  with multiplication  $\mu_0$ , denoted by  $\vartheta(\mu_0)$ , is the set of all its isomorphic algebras. The deformation techniques are used to do the geometric study of these varieties. The deformation attempts to understand which algebra we can get from the original one by deforming. At the same time it gives more informations about the structure of the algebra, for example we can look for which properties are stable under deformation.

The deformation of mathematical objects is one of the oldest techniques used by mathematicians. The different areas where the notion of deformation appears are geometry, complex manifolds (Kodaira and Spencer 1958, Kuranishi 1962), algebraic manifolds (Artin, Schlessinger 1968), Lie algebras (Nijenhuis, Richardson 1967) and rings and associative algebras (Gerstenhaber 1964). The dual notion of deformation (in some sense) is that of degeneration which appears first in physics literature (Segal 1951, Inonu and Wigner 1953). Degeneration is also called specialisation or contraction.

The quantum mechanics and quantum groups gave impulse to the theory of deformation. The theory of quantization was introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [3] to describe the quantum mechanics as a deformation of classical mechanics. It associates to a Poisson manifold a star-product, which is a one parameter family of associative algebras. In another hand the quantum groups are obtained by deforming the Hopf structure of an algebra, in particular enveloping algebras. In [4], we show the existence of associative deformation of an enveloping algebra using the linear Poisson structure of the Lie algebra.

In the following, we will recall first the different notions of deformation. The most used one is the formal deformation introduced by Gerstenhaber for rings and algebras [13], it uses a formal series and connects the theory of deformation to Hochschild cohomology. A noncommutative version, where the parameter no longer commutes with the element of algebra, was introduced by Pinzson [37] and generalized by Nadaud [33]. They describe the corresponding cohomology and show that the Weyl algebra, which is rigid for formal deformation, is nonrigid in the noncommutative case. In a Nonstandard framework, Goze introduced the notion of perturbation for studying the rigidity of Lie algebras [17][1], it was also used to describe the 6-dimensional rigid associative algebras [18][28]. The perturbation needs the concept of infinitely small elements, these elements are obtained here in an algebraic way. We construct an extension of real or complex numbers sets containing these elements. A more general notion called global deformation was introduced by Fialowski, following Schlessinger, for Lie algebras [7]. All these approaches are compared in section 3. The section 4 is devoted to universal and versal deformation. In Section 5, we describe the corresponding notions of degeneration with some general properties and examples. The Section 6 is devoted to the study of rigidity of algebra in each framework. The last section concerns the geometric study of the algebraic varieties  $Alg_n$  using the deformation tools.

In formal deformation the properties are described using the Hochschild cohomology groups. The global deformation seems to be the good framework to solve the problem of universal or versal deformations, the deformations which generate the others. Whereas, the perturbation approach is more adapted to direct computation.

## 2 The world of deformations

### 2.1 Gerstenhaber's formal deformation

Let  $\mathcal{A}$  be an associative algebra over a field  $\mathbb{K}$ ,  $V$  be the underlying vector space and  $\mu_0$  the multiplication.

Let  $\mathbb{K}[[t]]$  be the power series ring in one variable  $t$  and  $V[[t]]$  be the extension of  $V$  by extending the coefficient domain from  $\mathbb{K}$  to  $\mathbb{K}[[t]]$ . Then  $V[[t]]$  is a  $\mathbb{K}[[t]]$ -module and  $V[[t]] = V \otimes_{\mathbb{K}} \mathbb{K}[[t]]$ . Note that  $V$  is a submodule of  $V[[t]]$ . We can obtain an extension of  $V$  with a structure of vector space by extending the coefficient domain from  $\mathbb{K}$  to  $\mathbb{K}((t))$ , the quotient power series field of  $\mathbb{K}[[t]]$ .

Any bilinear map  $f : V \times V \rightarrow V$  (in particular the multiplication in  $\mathcal{A}$ ) can be extended to a bilinear map from  $V[[t]] \times V[[t]]$  to  $V[[t]]$ .

**Definition 1** Let  $\mu_0$  be the multiplication of the associative algebra  $\mathcal{A}$ . A deformation of  $\mu_0$  is a one parameter family  $\mu_t$  in  $\mathbb{K}[[t]] \otimes V$  over the formal power series ring  $\mathbb{K}[[t]]$  of the form  $\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \dots$  where  $\mu_i \in \text{Hom}(V \times V, V)$  (bilinear maps) satisfying the (formally) condition of associativity :

$$\text{For all } X, Y, Z \in V \quad \mu_t(\mu_t(X, Y), Z) = \mu_t(X, \mu_t(Y, Z))$$

We note that the deformation of  $\mathcal{A}$  is a  $\mathbb{K}$ -algebra structure on  $\mathcal{A}[[t]]$  such that  $\mathcal{A}[[t]]/t\mathcal{A}[[t]]$  is isomorphic to  $\mathcal{A}$ .

The previous equation is equivalent to an infinite equation system and it is called the *deformation equation*. The resolution of the deformation equation connects deformation theory to Hochschild cohomology. Let  $C^d(\mathcal{A}, \mathcal{A})$  be the space of  $d$ -cochains, the space of multilinear maps from  $V^{\times d}$  to  $V$ .

The boundary operator  $\delta_d$ , which we denote by  $\delta$  if there is no ambiguity :

$$\begin{aligned} \delta_d : \quad & C^d(\mathcal{A}, \mathcal{A}) \rightarrow C^{d+1}(\mathcal{A}, \mathcal{A}) \\ & \varphi \rightarrow \delta_d \varphi \end{aligned}$$

is defined for  $(x_1, \dots, x_{d+1}) \in V^{\times(d+1)}$  by

$$\begin{aligned} \delta_d \varphi(x_1, \dots, x_{d+1}) &= \mu_0(x_1, \varphi(x_2, \dots, x_{d+1})) + \\ & \sum_{i=1}^d (-1)^i \varphi(x_1, \dots, \mu_0(x_i, x_{i+1}), \dots, x_d) + (-1)^{d+1} \mu_0(\varphi(x_1, \dots, x_d), x_{d+1}) \end{aligned}$$

The group of  $d$ -cocycles is :  $Z^d(\mathcal{A}, \mathcal{A}) = \{\varphi \in C^d(\mathcal{A}, \mathcal{A}) / \delta^d \varphi = 0\}$

The group of  $d$ -coboundaries is

$$B^d(\mathcal{A}, \mathcal{A}) = \{\varphi \in C^d(\mathcal{A}, \mathcal{A}) / \varphi = \delta_{d-1} f, f \in C^{d-1}(\mathcal{A}, \mathcal{A})\}$$

The Hochschild cohomology group of the algebra  $\mathcal{A}$  with coefficient in itself is

$$H^d(\mathcal{A}, \mathcal{A}) = Z^d(\mathcal{A}, \mathcal{A}) / B^d(\mathcal{A}, \mathcal{A}).$$

We define two maps  $\circ$  and  $[ , ]_G$

$$\circ, [ , ]_G : C^d(\mathcal{A}, \mathcal{A}) \times C^e(\mathcal{A}, \mathcal{A}) \rightarrow C^{d+e-1}(\mathcal{A}, \mathcal{A})$$

by

$$(\varphi \circ \psi)(a_1, \dots, a_{d+e-1}) = \sum_{i \geq 0} (-1)^{i(e-1)} \varphi(a_1, \dots, a_i, \psi(a_{i+1}, \dots, a_{i+e}), \dots)$$

and

$$[\varphi, \psi]_G = \varphi \circ \psi - (-1)^{(e-1)(d-1)} \psi \circ \varphi.$$

The space  $(C(\mathcal{A}, \mathcal{A}), \circ)$  is a pre-Lie algebra and  $(C(\mathcal{A}, \mathcal{A}), [ , ]_G)$  is a graded Lie algebra (Lie superalgebra). The bracket  $[ , ]_G$  is called Gerstenhaber's bracket. The square of  $[\mu_0, \cdot]_G$  vanishes and defines the 2-coboundary operator. The multiplication  $\mu_0$  of  $\mathcal{A}$  is associative if  $[\mu_0, \mu_0]_G = 0$ .

Now, we discuss the deformation equation in terms of cohomology. The deformation equation may be written

$$(*) \quad \left\{ \sum_{i=0}^k \mu_i \circ \mu_{k-i} = 0 \quad k = 0, 1, 2, \dots \right.$$

The first equation ( $k = 0$ ) is the associativity condition for  $\mu_0$ . The second equation shows that  $\mu_1$  must be a 2-cocycle for Hochschild cohomology ( $\mu_1 \in Z^2(\mathcal{A}, \mathcal{A})$ ). More generally, suppose that  $\mu_p$  be the first non-zero coefficient after  $\mu_0$  in the deformation  $\mu_t$ . This  $\mu_p$  is called the *infinitesimal* of  $\mu_t$  and is a 2-cocycle of the Hochschild cohomology of  $\mathcal{A}$  with coefficient in itself.

*The cocycle  $\mu_p$  is called integrable if it is the first term, after  $\mu_0$ , of an associative deformation.*

The integrability of  $\mu_p$  implies an infinite sequence of relations which may be interpreted as the vanishing of the obstruction to the integration of  $\mu_p$ .

For an arbitrary  $k > 1$ , the  $k^{\text{th}}$  equation of the system  $(*)$  may be written

$$\delta \mu_k = \sum_{i=1}^{k-1} \mu_i \circ \mu_{k-i}$$

Suppose that the truncated deformation  $\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \dots + t^{m-1}\mu_{m-1}$  satisfies the deformation equation. The truncated deformation is extended to a deformation of order  $m$ , i.e.  $\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \dots + t^{m-1}\mu_{m-1} + t^m\mu_m$ , satisfying the deformation equation if

$$\delta \mu_m = \sum_{i=1}^{m-1} \mu_i \circ \mu_{m-i}$$

The right-hand side of this equation is called the *obstruction* to finding  $\mu_m$  extending the deformation.

*The obstruction is a Hochschild 3-cocycle. Then, if  $H^3(\mathcal{A}, \mathcal{A}) = 0$  it follows that all obstructions vanish and every  $\mu_m \in Z^2(\mathcal{A}, \mathcal{A})$  is integrable.*

Given two associative deformations  $\mu_t$  and  $\mu'_t$  of  $\mu_0$ , we say that they are equivalent if there is a formal isomorphism  $F_t$  which is a  $\mathbb{K}[[t]]$ -linear map that may be written in the form

$$F_t = Id + t f_1 + t^2 f_2 + \dots \quad \text{where } f_i \in \text{End}_{\mathbb{K}}(V)$$

such that  $\mu_t = F_t \cdot \mu'_t$  defined by

$$\mu_t(X, Y) = F_t^{-1}(\mu'_t((F_t(X), F_t(Y))) \quad \text{for all } X, Y \in V$$

A deformation  $\mu_t$  of  $\mu_0$  is called *trivial* if and only if  $\mu_t$  is equivalent to  $\mu_0$ .

**Proposition 2.1** *Every nontrivial deformation  $\mu_t$  of  $\mu_0$  is equivalent to  $\mu_t = \mu_0 + t^p \mu'_p + t^{p+1} \mu'_{p+1} + \dots$  where  $\mu'_p \in Z^2(\mathcal{A}, \mathcal{A})$  and  $\mu'_p \notin B^2(\mathcal{A}, \mathcal{A})$ .*

Then we have this fundamental and well-known theorem.

**Theorem 2.1** *If  $H^2(\mathcal{A}, \mathcal{A}) = 0$ , then all deformations of  $\mathcal{A}$  are equivalent to a trivial deformation.*

**Remark 2.1** *The formal deformation notion is extended to coalgebras and bialgebras in [15].*

## 2.2 Non-commutative formal deformation

In previous formal deformation the parameter commutes with the original algebra. Motivated by some nonclassical deformation appearing in quantization of Nambu mechanics, Pinczon introduced a deformation called noncommutative deformation where the parameter no longer commutes with the original algebra. He developed also the associated cohomology [37].

Let  $\mathcal{A}$  be a  $\mathbb{K}$ -vector space and  $\sigma$  be an endomorphism of  $\mathcal{A}$ . We give  $\mathcal{A}[[t]]$  a  $\mathbb{K}[[t]]$ -bimodule structure defined for every  $a_p \in \mathcal{A}, \lambda_q \in \mathbb{K}$  by :

$$\begin{aligned} \sum_{p \geq 0} a_p t^p \cdot \sum_{q \geq 0} \lambda_q t^q &= \sum_{p, q \geq 0} \lambda_q a_p t^{p+q} \\ \sum_{q \geq 0} \lambda_q t^q \cdot \sum_{p \geq 0} a_p t^p &= \sum_{p, q \geq 0} \lambda_q \sigma^q(a_p) t^{p+q} \end{aligned}$$

**Definition 2** *A  $\sigma$ -deformation of an algebra  $\mathcal{A}$  is a  $\mathbb{K}$ -algebra structure on  $\mathcal{A}[[t]]$  which is compatible with the previous  $\mathbb{K}[[t]]$ -bimodule structure and such that*

$$\mathcal{A}[[t]]/(\mathcal{A}[[t]]t) \cong \mathcal{A}.$$

The previous deformation was generalized by F. Nadaud [33] where he considered deformations based on two commuting endomorphisms  $\sigma$  and  $\tau$ . The  $\mathbb{K}[[t]]$ -bimodule structure on  $\mathcal{A}[[t]]$  is defined for  $a \in \mathcal{A}$  by the formulas  $t \cdot a = \sigma(a)t$  and  $a \cdot t = \tau(a)t$ , ( $a \cdot t$  being the right action of  $t$  on  $a$ ).

The remarkable difference with commutative deformation is that the Weyl algebra of differential operators with polynomial coefficients over  $\mathbb{R}$  is rigid for commutative deformations but has a nontrivial noncommutative deformation; it is given by the enveloping algebra of the Lie superalgebra  $osp(1, 2)$ .

## 2.3 Global deformation

The approach follows from a general fact in Schlessinger's works [39] and was developed by A. Fialowski [6]. She applies it to construct deformations of Witt Lie subalgebras. We summarize here the notion of global deformation in the case of associative algebra. Let  $\mathbb{B}$  be a commutative algebra over a field  $\mathbb{K}$  of characteristic 0 and augmentation morphism  $\varepsilon : \mathcal{A} \rightarrow \mathbb{K}$  (a  $\mathbb{K}$ -algebra homomorphism,  $\varepsilon(1_{\mathbb{B}}) = 1$ ). We set  $m_{\varepsilon} = \text{Ker}(\varepsilon)$ ;  $m_{\varepsilon}$  is a maximal ideal of  $\mathbb{B}$ . (A maximal ideal  $m$  of  $\mathbb{B}$  such that  $\mathcal{A}/m \cong \mathbb{K}$ , defines naturally an augmentation). We call  $(\mathbb{B}, m)$  base of deformation.

**Definition 3** A global deformation of base  $(\mathbb{B}, m)$  of an algebra  $\mathcal{A}$  with a multiplication  $\mu$  is a structure of  $\mathbb{B}$ -algebra on the tensor product  $\mathbb{B} \otimes_{\mathbb{K}} \mathcal{A}$  with a multiplication  $\mu_{\mathbb{B}}$  such that  $\varepsilon \otimes \text{id} : \mathbb{B} \otimes \mathcal{A} \rightarrow \mathbb{K} \otimes \mathcal{A} = \mathcal{A}$  is an algebra homomorphism. i.e.  $\forall a, b \in \mathbb{B}$  and  $\forall x, y \in \mathcal{A}$  :

1.  $\mu_{\mathbb{B}}(a \otimes x, b \otimes y) = (ab \otimes \text{id})\mu_{\mathbb{B}}(1 \otimes x, 1 \otimes y)$  ( $\mathbb{B}$ -linearity)
2. The multiplication  $\mu_{\mathbb{B}}$  is associative.
3.  $\varepsilon \otimes \text{id}(\mu_{\mathbb{B}}(1 \otimes x, 1 \otimes y)) = 1 \otimes \mu(x, y)$

**Remark 2.2** Condition 1 shows that to describe a global deformation it is enough to know the products  $\mu_{\mathbb{B}}(1 \otimes x, 1 \otimes y)$ , where  $x, y \in \mathcal{A}$ . The conditions 1 and 2 show that the algebra is associative and the last condition insures the compatibility with the augmentation. We deduce

$$\mu_{\mathbb{B}}(1 \otimes x, 1 \otimes y) = 1 \otimes \mu(x, y) + \sum_i \alpha_i \otimes z_i \quad \text{with } \alpha_i \in m, z_i \in \mathcal{A}$$

### 2.3.1 Equivalence and push-out

- A global deformation is called *trivial* if the structure of  $\mathbb{B}$ -algebra on  $\mathbb{B} \otimes_{\mathbb{K}} \mathcal{A}$  satisfies  $\mu_{\mathbb{B}}(1 \otimes x, 1 \otimes y) = 1 \otimes \mu(x, y)$ .
- Two deformations of an algebra with the same base are called *equivalent* (or isomorphic) if there exists an algebra isomorphism between the two copies of  $\mathbb{B} \otimes_{\mathbb{K}} \mathcal{A}$ , compatible with  $\varepsilon \otimes \text{id}$ .
- A global deformation with base  $(\mathbb{B}, m)$  is called *local* if  $\mathbb{B}$  is a local  $\mathbb{K}$ -algebra with a unique maximal ideal  $m_{\mathbb{B}}$ . If, in addition  $m_{\mathbb{B}}^2 = 0$ , the deformation is called *infinitesimal*.
- Let  $\mathbb{B}'$  be another commutative algebra over  $\mathbb{K}$  with augmentation  $\varepsilon' : \mathbb{B}' \rightarrow \mathbb{K}$  and  $\Phi : \mathbb{B} \rightarrow \mathbb{B}'$  an algebra homomorphism such that  $\Phi(1_{\mathbb{B}}) = 1_{\mathbb{B}'}$  and  $\varepsilon' \circ \Phi = \varepsilon$ . If a deformation  $\mu_{\mathbb{B}}$  with a base  $(\mathbb{B}, \text{Ker}(\varepsilon))$  of  $\mathcal{A}$  is given we call push-out  $\mu_{\mathbb{B}'} = \Phi_* \mu_{\mathbb{B}}$  a deformation of  $\mathcal{A}$  with a base  $(\mathbb{B}', \text{Ker}(\varepsilon'))$  with the following algebra structure on  $\mathbb{B}' \otimes \mathcal{A} = (\mathbb{B}' \otimes_{\mathbb{B}} \mathbb{B}) \otimes \mathcal{A} = \mathbb{B}' \otimes_{\mathbb{B}} (\mathbb{B} \otimes \mathcal{A})$

$$\mu_{\mathbb{B}'}(a'_1 \otimes_{\mathbb{B}} (a_1 \otimes x_1), a'_2 \otimes_{\mathbb{B}} (a_2 \otimes x_2)) := a'_1 a'_2 \otimes_{\mathbb{B}} \mu_{\mathbb{B}}(a_1 \otimes x_1, a_2 \otimes x_2)$$

with  $a'_1, a'_2 \in \mathbb{B}', a_1, a_2 \in \mathbb{B}, x_1, x_2 \in \mathcal{A}$ . The algebra  $\mathbb{B}'$  is viewed as a  $\mathbb{B}$ -module with the structure  $aa' = a'\Phi(a)$ . Suppose that

$$\mu_{\mathbb{B}}(1 \otimes x, 1 \otimes y) = 1 \otimes \mu(x, y) + \sum_i \alpha_i \otimes z_i$$

with  $\alpha_i \in m, z_i \in \mathcal{A}$ , then

$$\mu_{\mathbb{B}'}(1 \otimes x, 1 \otimes y) = 1 \otimes \mu(x, y) + \sum_i \Phi(\alpha_i) \otimes z_i$$

### 2.3.2 Coalgebra and Hopf algebra global deformation

The global deformation may be extended to coalgebra structures, then to Hopf algebras. Let  $\mathcal{C}$  be a coalgebra over  $\mathbb{K}$ , defined by the comultiplication  $\Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$ . Let  $\mathbb{B}$  be a commutative algebra over  $\mathbb{K}$  and let  $\varepsilon$  be an augmentation  $\varepsilon : \mathbb{B} \rightarrow \mathbb{K}$  with  $m = \text{Ker}(\varepsilon)$  a maximal ideal .

A global deformation with base  $(\mathbb{B}, m)$  of coalgebra  $\mathcal{C}$  with a comultiplication  $\Delta$  is a structure of  $\mathbb{B}$ -coalgebra on the tensor product  $\mathbb{B} \otimes_{\mathbb{K}} \mathcal{C}$  with the comultiplication  $\Delta_{\mathbb{B}}$  such that  $\varepsilon \otimes \text{id} : \mathbb{B} \otimes \mathcal{C} \rightarrow \mathbb{K} \otimes \mathcal{C} = \mathcal{C}$  is a coalgebra homomorphism. i.e  $\forall a \in \mathbb{B}$  and  $\forall x \in \mathcal{C}$  :

1.  $\Delta_{\mathbb{B}}(a \otimes x) = a\Delta_{\mathbb{B}}(1 \otimes x)$
2. The comultiplication  $\Delta_{\mathbb{B}}$  is coassociative.
3.  $(\varepsilon \otimes \text{id}) \otimes (\varepsilon \otimes \text{id}) (\Delta_{\mathbb{B}}(1 \otimes x)) = 1 \otimes \Delta(x)$

The comultiplication  $\Delta_{\mathbb{B}}$  may be written for all  $x$  as :

$$\Delta_{\mathbb{B}}(1 \otimes x) = 1 \otimes \Delta(x) + \sum_i \alpha_i \otimes z_i \otimes z'_i \quad \text{with } \alpha_i \in m, z_i, z'_i \in \mathcal{C}$$

### 2.3.3 Equivalence and push-out for coalgebras

Two global deformations of a coalgebra with the same base are called *equivalent* (or isomorphic) if there exists a coalgebra isomorphism between the two copies of  $\mathbb{B} \otimes_{\mathbb{K}} \mathcal{C}$ , compatible with  $\varepsilon \otimes \text{id}$ .

Let  $\mathbb{B}'$  be another commutative algebra over  $\mathbb{K}$  as in section 2.3.1. If  $\Delta_{\mathbb{B}}$  is a global deformation with a base  $(\mathbb{B}, \text{Ker}(\varepsilon))$  of a coalgebra  $\mathcal{C}$ . We call push-out  $\Delta_{\mathbb{B}'} = \Phi_* \Delta_{\mathbb{B}}$  a global deformation with a base  $(\mathbb{B}', \text{Ker}(\varepsilon'))$  of  $\mathcal{C}$  with the following coalgebra structure on  $\mathbb{B}' \otimes \mathcal{C} = (\mathbb{B}' \otimes_{\mathbb{B}} \mathbb{B}) \otimes \mathcal{C} = \mathbb{B}' \otimes_{\mathbb{B}} (\mathbb{B} \otimes \mathcal{C})$

$$\Delta_{\mathbb{B}'}(a' \otimes_{\mathbb{B}} (a \otimes x)) := a' \otimes_{\mathbb{B}} \Delta_{\mathbb{B}}(a \otimes x)$$

with  $a' \in \mathbb{B}', a \in \mathbb{B}, x \in \mathcal{C}$ . The algebra  $\mathbb{B}'$  is viewed as a  $\mathbb{B}$ -module with the structure  $aa' = a'\Phi(a)$ . Suppose that

$$\Delta_{\mathbb{B}}(1 \otimes x) = 1 \otimes \Delta(x) + \sum_i \alpha_i \otimes z_i \otimes z'_i$$

with  $\alpha_i \in m, z_i \in \mathcal{C}$ , then

$$\Delta_{\mathbb{B}'}(1 \otimes x) = 1 \otimes \Delta(x) + \sum_i \Phi(\alpha_i) \otimes z_i \otimes z'_i$$

### 2.3.4 Hopf algebra global deformation

Naturally, we can define Hopf algebra global deformation from the algebra and coalgebra global deformation.

### 2.3.5 Valued global deformation.

In [20], Goze and Remm considered the case where the base algebra  $\mathbb{B}$  is a commutative  $\mathbb{K}$ -algebra of valuation such that the residual field  $\mathbb{B}/m$  is isomorphic to  $\mathbb{K}$ , where  $m$  is the maximal ideal (recall that  $\mathbb{B}$  is a valuation ring of a field  $F$  if  $\mathbb{B}$  is a local integral domain satisfying  $x \in F - \mathbb{B}$  implies  $x^{-1} \in m$ ). They called such deformations valued global deformations.

## 2.4 Perturbation theory

The perturbation theory over complex numbers is based on an enlargement of the field of real numbers with the same algebraic order properties as  $\mathbb{R}$ . The extension  $\mathbb{R}^*$  of  $\mathbb{R}$  induces the existence of infinitesimal element in  $\mathbb{R}^*$ , hence in  $(\mathbb{C}^n)^*$ . The "infinitely small number" and "illimited number" have a long historical tradition (Euclid, Eudoxe, Archimedes..., Cavalieri, Galilei...Leibniz, Newton). The infinitesimal methods was considered in heuristic and intuitive way until 1960 when A. Robinson gave a rigorous foundation to these methods [38]. He used methods of mathematical logic, he constructed a NonStandard model for real numbers. However, there exists other frames for infinitesimal methods (see [34], [24], [25]). In order to study the local properties of complex Lie algebras M. Goze introduced in 1980 the notion of perturbation of algebraic structure (see [17]) in Nelson's framework [34]. The description given here is more algebraic, it is based on Robinson's framework. First, we summarize a description of the hyperreals field and then we define the perturbation notion over hypercomplex numbers.

### 2.4.1 Field of hyperreals and their properties

The construction of hyperreal numbers system needs four axioms. They induce a triple  $(\mathbb{R}, \mathbb{R}^*, \star)$  where  $\mathbb{R}$  is the real field,  $\mathbb{R}^*$  is the hyperreal field and  $\star$  is a natural mapping.

Axiom 1.  $\mathbb{R}$  is an archimedean ordered field.

Axiom 2.  $\mathbb{R}^*$  is a proper ordered field extension of  $\mathbb{R}$ .

Axiom 3. For each real function  $f$  of  $n$  variables there is a corresponding hyperreal function  $f^*$  of  $n$  variables, called natural extension of  $f$ . The field operations of  $\mathbb{R}^*$  are the natural extensions of the field operations of  $\mathbb{R}$ .

Axiom 4. If two systems of formulae have exactly the same real solutions, they have exactly the same hyperreal solutions.

The following theorem shows that such extension exists

**Theorem 2.2** *Let  $\mathbb{R}$  be the ordered field of real numbers. There is an ordered field extension  $\mathbb{R}^*$  of  $\mathbb{R}$  and a mapping  $\star$  from real functions to hyperreal functions such that the axioms 1-4 hold.*

In the proof, the plan is to find an infinite set  $M$  of formulas  $\varphi(x)$  which describe all properties of a positive infinitesimal  $x$ , and to built  $\mathbb{R}^*$  out of this set of formulas. See [23] pp 23-24 for a complete proof.

**Definition 4** •  $x \in \mathbb{R}^*$  is called *infinitely small* or *infinitesimal* if  $|x| < r$  for all  $r \in \mathbb{R}^+$ .

- $x \in \mathbb{R}^*$  is called *limited* if there exists  $r \in \mathbb{R}$  such that  $|x| < r$ .
- $x \in \mathbb{R}^*$  is called *illimited* or *infinitely large* if  $|x| > r$  for all  $r \in \mathbb{R}$
- Two elements  $x$  and  $y$  of  $\mathbb{R}^*$  are called *infinitely close* ( $x \simeq y$ ) if  $x - y$  is infinitely small.

**Definition 5** Given a hyperreal number  $x \in \mathbb{R}^*$ , we set

$$\text{halo}(x) = \{y \in \mathbb{R}^*, \quad x \simeq y\}$$

$$\text{galaxy}(x) = \{y \in \mathbb{R}^*, \quad x - y \text{ is limited}\}$$

In particular,  $\text{halo}(0)$  is the set of infinitely small elements of  $\mathbb{R}^*$  and  $\text{galaxy}(0)$  is the set of limited hyperreal numbers which we denote by  $\mathbb{R}_L^*$ .

**Proposition 2.2** 1.  $\mathbb{R}_L^*$  and  $\text{halo}(0)$  are subrings of  $\mathbb{R}^*$ .

2.  $\text{halo}(0)$  is a maximal ideal in  $\mathbb{R}_L^*$ .

**Proof 1** The first property is easy to prove. (2) Let  $\eta \simeq 0$  and  $a \in \mathbb{R}$ , then there exists  $t \in \mathbb{R}$  such that  $|a| < t$  and for all  $r \in \mathbb{R}$ ,  $|\eta| < r/t$ , thus  $|a\eta| < r$ . Therefore,  $a\eta \in \mathbb{R}_L^*$  and  $\text{halo}(0)$  is an ideal in  $\mathbb{R}_L^*$ . Let us show that the ideal is maximal. We note that  $x \in \mathbb{R}^*$  is illimited is equivalent to  $x^{-1}$  is infinitely small. Assume that there exists an ideal  $I$  in  $\mathbb{R}_L^*$  containing  $\text{halo}(0)$  and let  $a \in I \setminus \text{halo}(0)$ , we have  $a^{-1} \in \mathbb{R}_L^*$ , thus  $1 = aa^{-1} \in I$ , so  $I \equiv \mathbb{R}_L^*$ .

The following theorem shows that there is a homomorphism of ring  $\mathbb{R}_L^*$  onto the field of real numbers.

**Theorem 2.3** Every  $x$  of  $\mathbb{R}_L^*$  admits a unique  $x_0$  in  $\mathbb{R}$  (called *standard part* of  $x$  and denoted by  $st(x)$ ) such that  $x \simeq x_0$ .

**Proof 2** Let  $x \in \mathbb{R}_L^*$ , assume that it exists two real numbers  $r$  and  $s$  such that  $x \simeq r$  and  $x \simeq s$ , this implies that  $r - s \simeq 0$ . Since the unique infinitesimal real number is 0 then  $r = s$ . To show the existence, set  $X = \{s \in \mathbb{R} : s < x\}$ . Then  $X \neq \emptyset$  and  $X < r$ , where  $r$  is a positive real number ( $-r < x < r$ ). Let  $t$  be the smallest  $r$ , for all positive real  $r$ , we have  $x \leq t + r$ , then  $x - t \leq r$  and  $t - r < x$ , it follows  $-(x - t) \leq r$ . Then  $x - t \simeq 0$ , hence  $x \simeq t$ .

We check easily that for  $x, y \in \mathbb{R}_L^*$ ,  $st(x + y) = st(x) + st(y)$ ,  $st(x - y) = st(x) - st(y)$  and  $st(xy) = st(x)st(y)$ . Also, for all  $r \in \mathbb{R}$ ,  $st(r) = r$ .

**Remark 2.3** All these concepts and properties may be extended to hyperreal vectors of  $\mathbb{R}^{*n}$ . An element  $v = (x_1, \dots, x_n)$ , where  $x_i \in \mathbb{R}^*$ , is called *infinitely small* if  $|v| = \sqrt{\sum_{i=1}^n x_i^2}$  is an infinitely small hyperreal and called *limited* if  $|v|$  is limited. Two vectors  $\mathbb{R}^{*n}$  are *infinitely close* if their difference is infinitely small. The extension to complex numbers and vectors is similar ( $\mathbb{C}^* = \mathbb{R}^* \times \mathbb{R}^*$ ).

### 2.4.2 Perturbation of associative algebras.

We introduce here the perturbation notion of an algebraic structure. Let  $V = \mathbb{K}^n$  be a  $\mathbb{K}$ -vector space ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ) and  $V^*$  be a vector space over  $\mathbb{K}^*$  ( $\mathbb{K}^* = \mathbb{R}^*$  or  $\mathbb{C}^*$ ). Let  $\mathcal{A}$  be an associative algebra of  $Alg_n$  with a multiplication  $\mu_0$  over  $V$ .

**Definition 6** A morphism  $\mu$  in  $Hom((V^*)^{\times 2}, V^*)$  is a perturbation of  $\mu_0$  if

$$\forall X_1, X_2 \in V : \mu(X_1, X_2) \simeq \mu_0(X_1, X_2)$$

and  $\mu$  satisfies the associativity condition over  $V$ . We write  $(\mu \simeq \mu_0)$ .

Fixing a basis of  $V$ , the extension  $V^*$  is a vector space which may be taken with the same basis as  $V$ . Then  $\mu$  is a perturbation of  $\mu_0$  is equivalent to say that the difference between the structure constants of  $\mu$  and  $\mu_0$  is an infinitesimal vector in  $V^*$ .

Two perturbations  $\mu$  and  $\mu'$  of  $\mu_0$  are isomorphic if there is an invertible map  $f$  in  $Hom(V^*, V^*)$  such that  $\mu' = f^{-1} \circ \mu \circ (f \otimes f)$ .

The following decomposition of a perturbation follows from Goze's decomposition [17].

**Theorem 2.4** Let  $\mathcal{A}$  be an algebra in  $Alg_n$  with a multiplication  $\mu_0$  and let  $\mu$  be a perturbation of  $\mu_0$ . We have the following decomposition of  $\mu$  :

$$\mu = \mu_0 + \varepsilon_1 \varphi_1 + \varepsilon_1 \varepsilon_2 \varphi_2 + \dots + \varepsilon_1 \cdots \varepsilon_k \varphi_k$$

where

1.  $\varepsilon_1, \dots, \varepsilon_k$  are non zero infinitesimals in  $\mathbb{K}^*$ .
2.  $\varphi_1, \dots, \varphi_k$  are independent bilinear maps in  $Hom(V^{\times 2}, V)$ .

**Remark 2.4** 1. The integer  $k$  is called the length of the perturbation. It satisfies  $k \leq n^3$ .

2. The perturbation decomposition is generalized in the valued global deformation case [20] by taking the  $\varepsilon_i$  in maximal ideal of a valuation ring.
3. The associativity of  $\mu$  is equivalent to a finite system of equation called the perturbation equation. This equation is studied above by using Massey cohomology products .

### 2.4.3 Resolution of the perturbation equation

In the following we discuss the conditions on  $\varphi_1$  such that it is a first term of a perturbation.

Let us consider a perturbation of length 2,  $\mu = \mu_0 + \varepsilon_1 \varphi_1 + \varepsilon_1 \varepsilon_2 \varphi_2$ , the perturbation equation is equivalent to

$$\begin{cases} \delta \varphi_1 = 0 \\ \varepsilon_1 [\varphi_1, \varphi_1]_G + 2\varepsilon_1 \varepsilon_2 [\varphi_1, \varphi_2]_G + \varepsilon_1 \varepsilon_2^2 [\varphi_2, \varphi_2]_G + 2\varepsilon_2 \delta \varphi_2 = 0 \end{cases}$$

Where  $[\varphi_i, \varphi_j]_G$  is the trilinear map defined by the Gerstenhaber bracket (see section 2.1) :

It follows, as in the deformation equation, that  $\varphi_1$  is a cocycle of  $Z^2(\mathcal{A}, \mathcal{A})$ . The second equation has infinitesimal coefficients but the vectors

$$\{[\varphi_1, \varphi_1]_G, [\varphi_1, \varphi_2]_G, [\varphi_2, \varphi_2]_G, \delta\varphi_2\}$$

are in  $\text{Hom}(V^{\times 3}, V)$  and form a system of rank 1.

**Proposition 2.3** *Let  $\mu$  be a perturbation of length  $k$  of the multiplication  $\mu_0$ . The rank of the vectors  $\{[\varphi_i, \varphi_j]_G, \delta\varphi_i\}$   $i = 1, \dots, k$  and  $i \leq j \leq k$  is equal to the rank of the vectors  $\{[\varphi_i, \varphi_j]_G\}$   $i = 1, \dots, k-1$  and  $i \leq j \leq k-1$ .*

**Proof 3** *We consider a non trivial linear form  $\omega$  containing in its kernel  $\{[\varphi_i, \varphi_j]_G\}$   $i = 1, \dots, k-1$  and  $i \leq j \leq k-1$ . We apply it to the perturbation equation then it follows that all the vectors  $\{[\varphi_i, \varphi_j]_G, \delta\varphi_i\}$   $i = 1, \dots, k$  and  $i \leq j \leq k$  are in the kernel of  $\omega$ .*

The following theorem, which uses the previous proposition, characterizes the cocycle which should be a first term of a perturbation, see [26] for the proof and [30] for the Massey products.

**Theorem 2.5** *Let  $\mathcal{A}$  be an algebra in  $\text{Alg}_n$  with a multiplication  $\mu_0$ . A vector  $\varphi_1$  in  $Z^2(\mathcal{A}, \mathcal{A})$  is the first term of a perturbation  $\mu$  of  $\mu_0$  ( $k$  being the length of  $\mu$ ) if and only if*

1. *The massey products  $[\varphi_1^2], [\varphi_1^3] \dots [\varphi_1^p]$  vanish until  $p = k^2$ .*
2. *The product representatives in  $B^3(\mathcal{A}, \mathcal{A})$  form a system of rank less or equal to  $k(k-1)/2$ .*

## 2.5 Coalgebra perturbations

The perturbation notion can be generalized to any algebraic structure on  $V$ . A morphism  $\mu$  in  $\text{Hom}((V^*)^{\times p}, (V^*)^{\times q})$  is a perturbation of  $\mu_0$  in  $\text{Hom}(V^{\times p}, V^{\times q})$  ( $\mu \simeq \mu_0$ ) if

$$\forall X_1, \dots, X_p \in V : \mu(X_1, \dots, X_p) \simeq \mu_0(X_1, \dots, X_p)$$

In particular, if  $p = 1$  and  $q = 2$ , we have the concept of coalgebra perturbation.

## 3 Comparison of the deformations

We show that the formal deformation and the perturbation are global deformations with appropriate bases, and we show that over complex numbers the perturbations contains all the convergent deformations.

**Global deformation and formal deformation** The following proposition gives the link between formal deformation and global deformation.

**Proposition 3.1** *Every formal deformation is a global deformation.*

Every formal deformation of an algebra  $\mathcal{A}$ , in Gerstenhaber sense, is a global deformation with a basis  $(\mathbb{B}, m)$  where  $\mathbb{B} = \mathbb{K}[[t]]$  and  $m = t\mathbb{K}[[t]]$ .

**Global deformation and perturbation** Let  $\mathbb{K}^*$  be a proper extension of  $\mathbb{K}$  described in the section (2.4) , where  $\mathbb{K}$  is the field  $\mathbb{R}$  or  $\mathbb{C}$ .

**Proposition 3.2** *Every perturbation of an algebra over  $\mathbb{K}$  is a global deformation with a base  $(\mathbb{K}_L^*, \text{halo}(0))$ .*

**Proof 4** *The set  $\mathbb{B} := \mathbb{K}_L^*$  is a local ring formed by the limited elements of  $\mathbb{K}^*$ . We define the augmentation  $\varepsilon : \mathbb{K}_L^* \rightarrow \mathbb{K}$  which associates to any  $x$  its standard part  $st(x)$ . The Kernel of  $\varepsilon$  corresponds to  $\text{halo}(0)$ , the set of infinitesimal elements of  $\mathbb{K}^*$ , which is a maximal ideal in  $\mathbb{K}_L^*$ .*

The multiplication  $\mu_B$  of a global deformation with a base  $(\mathbb{K}_L^*, \text{halo}(0))$  of an algebra with a multiplication  $\mu$  may be written, for all  $x, y \in \mathbb{K}^n$  :

$$\mu_B(1 \otimes x, 1 \otimes y) = 1 \otimes \mu(x, y) + \sum_i \alpha_i \otimes z_i \quad \text{avec } \alpha_i \in \text{halo}(0) \subset \mathbb{K}^*, \quad z_i \in \mathbb{K}^n$$

This is equivalent to say that  $\mu_B$  is a perturbation of  $\mu$ .

**Remark 3.1** *The corresponding global deformations of a perturbations are local but not infinitesimal because  $\text{halo}(0)^n \neq 0$ .*

### Formal deformation and perturbation

**Proposition 3.3** *Let  $\mathcal{A}$  be a  $\mathbb{C}$ -algebra in  $\text{Alg}_n$  with a multiplication  $\mu_0$ . Let  $\mu_t$  be a convergent deformation of  $\mu_0$  and  $\alpha$  be infinitesimal in  $\mathbb{C}^*$  then  $\mu_\alpha$  is a perturbation of  $\mu_0$ .*

**Proof 5** *Since the deformation is convergent then  $\mu_\alpha$  corresponds to a point of  $\text{Alg}_n \subset \mathbb{C}^{n^3}$  which is infinitely close to the corresponding point of  $\mu_0$ . Then it determines a perturbation of  $\mu_0$ .*

A perturbation should correspond to a formal deformation if we consider the power series ring  $\mathbb{K}[[t_1, \dots, t_r]]$  with more than one parameter.

## 4 Universal and versal deformation

Given an algebra, the problem is to find particular deformations which induces all the others in the space of all deformations or in a fixed category of deformations. We say that the deformation is universal if there is unicity of the homomorphism between base algebras. This problem is too hard in general but the global deformation seems more adapted to construct universal or versal deformation. This problem was considered in Lie algebras case for the categories of deformations over infinitesimal local algebras and complete local algebras (see [7], [9], [11]). They show that if we consider the infinitesimal deformations, i.e. the deformations over local algebras  $B$  such that  $m_B^2 = 0$  where  $m_B$  is the maximal ideal, then there exists a universal deformation. If we consider the category of complete local rings, then there does not exist a universal deformation but only versal deformation.

**Formal global deformation.** Let  $\mathbb{B}$  be a complete local algebra over  $\mathbb{K}$ , so  $\mathbb{B} = \varprojlim_{n \rightarrow \infty} (\mathbb{B}/m^n)$  (inductive limit), where  $m$  is the maximal ideal of  $\mathbb{B}$  and we assume that  $\mathbb{B}/m \cong \mathbb{K}$ .

**Definition 7** A formal global deformation of  $\mathcal{A}$  with base  $(\mathbb{B}, m)$  is an algebra structure on the completed tensor product  $\mathbb{B} \hat{\otimes} \mathcal{A} = \varprojlim_{n \rightarrow \infty} ((\mathbb{B}/m^n) \otimes \mathcal{A})$  such that  $\varepsilon \hat{\otimes} id : \mathbb{B} \hat{\otimes} \mathcal{A} \rightarrow K \otimes \mathcal{A} = \mathcal{A}$  is an algebra homomorphism.

**Remark 4.1** • The formal global deformation of  $\mathcal{A}$  with base  $(\mathbb{K}[[t]], t\mathbb{K}[[t]])$  are the same as formal one parameter deformation of Gerstenhaber.

- The perturbation are complete global deformation because  $\lim_{n \rightarrow \infty} \mathbb{K}^*/\text{halo}(0)^n$  is isomorphic to  $\mathbb{K}^*$  and  $\mathbb{K}^*$  is isomorphic  $\mathbb{K}$  as algebra.

We assume now that the algebra  $\mathcal{A}$  satisfies  $\dim(H^2(\mathcal{A}, \mathcal{A})) < \infty$ . We consider  $\mathbb{B} = \mathbb{K} \oplus H^2(\mathcal{A}, \mathcal{A})^{\text{dual}}$ . The following theorems du to Fialowski and Post [7][9] show the existence of universal infinitesimal deformation under the previous assumptions.

**Theorem 4.1** There exists, in the category of infinitesimal global deformations, a universal infinitesimal deformation  $\eta_{\mathcal{A}}$  with base  $\mathbb{B}$  equipped with the multiplication  $(\alpha_1, h_1) \cdot (\alpha_1, h_1) = (\alpha_1 \alpha_2, \alpha_1 h_2 + \alpha_2 h_1)$ .

Let  $\mathbb{P}$  be any finitedimensional local algebra over  $\mathbb{K}$ . The theorem means that for any infinitesimal deformation defined by  $\mu_{\mathbb{P}}$  of an algebra  $\mathcal{A}$ , there exists a unique homomorphism  $\Phi : \mathbb{K} \oplus H^2(\mathcal{A}, \mathcal{A})^{\text{dual}} \rightarrow \mathbb{P}$  such that  $\mu_{\mathbb{P}}$  is equivalent to the push-out  $\Phi_* \eta_{\mathcal{A}}$ .

**Definition 8** A formal global deformation  $\eta$  of  $\mathcal{A}$  parameterized by a complete local algebra  $\mathbb{B}$  is called versal if for any deformation  $\lambda$  of  $\mathcal{A}$ , parameterized by a complete local algebra  $(\mathbb{A}, m_{\mathbb{A}})$ , there is a morphism  $f : \mathbb{B} \rightarrow \mathbb{A}$  such that

- (1) The push-out  $f_* \eta$  is equivalent to  $\lambda$ .
- (2) If  $\mathbb{A}$  satisfies  $m_{\mathbb{A}}^2 = 0$ , then  $f$  is unique;

**Theorem 4.2** Let  $\mathcal{A}$  be an algebra .

- (1) There exists a versal formal global deformation of  $\mathcal{A}$ .
- (2) The base of the versal formal deformation is formally embedded into  $H^2(\mathcal{A}, \mathcal{A})$  (it can be described in  $H^2(\mathcal{A}, \mathcal{A})$  by a finite system of formal equations)

**Examples.** The Witt algebra is the infinite dimensional Lie algebra of polynomial vector fields spanned by the fields  $e_i = z^{i+1} \frac{d}{dz}$  with  $i \in \mathbb{Z}$ . In [7], Fialowski constructed versal deformation of the Lie subalgebra  $L_1$  of Witt algebra ( $L_1$  is spanned by  $e_n$ ,  $n > 0$ , while the bracket is given by  $[e_n, e_m] = (m - n)e_{n+m}$  ).

Here is three real deformations of the Lie algebra  $L_1$  which are non trivial and pairwise nonisomorphic.

$$\begin{aligned} [e_i, e_j]_t^1 &= (j - i)(e_{i+j} + te_{i+j-1}) \\ [e_i, e_j]_t^2 &= \begin{cases} (j - i)e_{i+j} & \text{if } i, j > 1 \\ (j - i)e_{i+j} + tje_j & \text{if } i = 1 \end{cases} \end{aligned}$$

$$[e_i, e_j]_t^3 = \begin{cases} (j-i)e_{i+j} & \text{if } i, j \neq 2 \\ (j-i)e_{i+j} + tje_j & \text{if } i = 2 \end{cases}$$

Fialowski and Post, in [9], study the  $L_2$  case. A more general procedure using the Harrisson cohomology of the commutative algebra  $\mathbb{B}$  is described by Fialowski and Fuchs in [10].

The Fialowski's global deformation of  $L_1$  can be realized as a perturbation, the parameter  $t_1, t_2$  and  $t_3$  have to be different infinitesimals in  $\mathbb{C}^*$ .

## 5 Degenerations

The notion of degeneration is fundamental in the geometric study of  $Alg_n$  and helps, in general, to construct new algebras.

**Definition 9** Let  $\mathcal{A}_0$  and  $\mathcal{A}_1$  be two  $n$ -dimensional algebras. We say that  $\mathcal{A}_0$  is a degeneration of  $\mathcal{A}_1$  if  $\mathcal{A}_0$  belongs to  $\overline{\vartheta(\mathcal{A}_1)}$ , the Zariski closure of the orbit of  $\mathcal{A}_1$ .

In the following we define degenerations in the different frameworks.

### 5.1 Global viewpoint

A characterization of global degeneration was given by Grunewald and O'Halloran in [21] :

**Theorem 5.1** Let  $\mathcal{A}_0$  and  $\mathcal{A}_1$  be two  $n$ -dimensional associative algebras over  $\mathbb{K}$  with the multiplications  $\mu_0$  and  $\mu_1$ . The algebra  $\mathcal{A}_0$  is a degeneration of  $\mathcal{A}_1$  if and only if there is a discrete valuation  $\mathbb{K}$ -algebra  $\mathbb{B}$  with residue field  $\mathbb{K}$  whose quotient field  $\mathcal{K}$  is finitely generated over  $\mathbb{K}$  of transcendence degree one (one parameter), and there is an  $n$ -dimensional algebra  $\mu_{\mathbb{B}}$  over  $\mathbb{B}$  such that  $\mu_{\mathbb{B}} \otimes \mathcal{K} \cong \mu_1 \otimes \mathcal{K}$  and  $\mu_{\mathbb{B}} \otimes \mathbb{K} \cong \mu_0$ .

### 5.2 Formal viewpoint

Let  $t$  be a parameter in  $\mathbb{K}$  and  $\{f_t\}_{t \neq 0}$  be a continuous family of invertible linear maps on  $V$  over  $\mathbb{K}$  and  $\mathcal{A}_1 = (V, \mu_1)$  be an algebra over  $\mathbb{K}$ . The limit (when it exists) of a sequence  $f_t \cdot \mathcal{A}_1$ ,  $\mathcal{A}_0 = \lim_{t \rightarrow 0} f_t \cdot \mathcal{A}_1$ , is a *formal degeneration* of  $\mathcal{A}_1$  in the sense that  $\mathcal{A}_0$  is in the Zariski closure of the set  $\{f_t \cdot \mathcal{A}_1\}_{t \neq 0}$ .

The multiplication  $\mu_0$  is given by

$$\mu_0 = \lim_{t \rightarrow 0} f_t \cdot \mu_1 = \lim_{t \rightarrow 0} f_t^{-1} \circ \mu_1 \circ f_t \times f_t$$

- The multiplication  $\mu_t = f_t^{-1} \circ \mu_1 \circ f_t \times f_t$  satisfies the associativity condition. Thus, when  $t$  tends to 0 the condition remains satisfied.
- The linear map  $f_t$  is invertible when  $t \neq 0$  and may be singular when  $t = 0$ . Then, we may obtain by degeneration a new algebra.
- The definition of formal degeneration maybe extended naturally to infinite dimensional case.

- When  $\mathbb{K}$  is the complex field, the multiplication given by the limit, follows from a limit of the structure constants, using the metric topology. In fact,  $f_t \cdot \mu$  corresponds to a change of basis when  $t \neq 0$ . When  $t = 0$ , they give eventually a new point in  $Alg_n \subset \mathbb{K}^{n^3}$ .
- If  $f_t$  is defined by a power serie the images over  $V \times V$  of the multiplication of  $f_t \cdot \mathcal{A}$  are in general in the Laurent power series ring  $V [[t, t^{-1}]]$ . But when the degeneration exists, it lies in the power series ring  $V [[t]]$ .

**Proposition 5.1** *Every formal degeneration is a global degeneration.*

The proof follows from the theorem (5.1) and the last remark.

### 5.3 Contraction

The notion of degeneration over the hypercomplex field is called contraction. It is defined by :

**Definition 10** *Let  $\mathcal{A}_0$  and  $\mathcal{A}_1$  be two algebras in  $Alg_n$  with multiplications  $\mu_0$  and  $\mu_1$ . The algebra  $\mathcal{A}_0$  is a contraction of  $\mathcal{A}_1$  if there exists a perturbation  $\mu$  of  $\mu_0$  such that  $\mu$  is isomorphic to  $\mu_1$ .*

The definition gives a characterization over hypercomplex field of  $\mu_0$  in the closure of the orbit of  $\mu_1$  (for the usual topology of  $\mathbb{C}^{n^3}$ ).

### 5.4 Examples

1. *The null algebra of  $Alg_n$  is a degeneration of every algebra of  $Alg_n$ .*

In fact, the null algebra is given in a basis  $\{e_1, \dots, e_n\}$  by the following non-trivial products  $\mu_0(e_1, e_i) = \mu_0(e_i, e_1) = e_i \quad i = 1, \dots, n$ .

Let  $\mu_1$  be a multiplication of any algebra of  $Alg_n$ , we have  $\mu_0 = \lim_{t \rightarrow 0} f_t \cdot \mu_1$  with  $f_t$  given by the diagonal matrix  $(1, t, \dots, t)$ .

For  $i \neq 1$  and  $j \neq 1$  we have

$$\begin{aligned} f_t \cdot \mu(e_i, e_j) &= f_t^{-1}(\mu(f_t(e_i), f_t(e_j))) = f_t^{-1}(\mu(te_i, te_j)) = \\ &= t^2 C_{ij}^1 e_1 + t \sum_{k>1} C_{ij}^k e_k \rightarrow 0 \text{ (when } t \rightarrow 0) \end{aligned}$$

For  $i = 1$  and  $j \neq 1$  we have

$$f_t \cdot \mu(e_1, e_j) = f_t^{-1}(\mu(f_t(e_1), f_t(e_j))) = f_t^{-1}(\mu(e_1, te_j)) = f_t^{-1}(te_j) = e_j.$$

This shows that every algebra of  $Alg_n$  degenerates formally to a null algebra.

By taking the parameter  $t = \alpha$ ,  $\alpha$  infinitesimal in the field of hypercomplex numbers, we get that the null algebra is a contraction of any complex associative algebra in  $Alg_n$ . In fact,  $f_\alpha \cdot \mu_1$  is a perturbation of  $\mu_0$  and is isomorphic to  $\mu_1$ .

The same holds also in the global viewpoint : Let  $B = \mathbb{K}[t]_{<t>}$  be the polynomial ring localized at the prime ideal  $< t >$  and let  $\mu_B = t\mu_1$  on the elements different from the unity and  $\mu_B = \mu_1$  elsewhere. Then  $\mu_1$  is  $\mathbb{K}(t)$ -isomorphic to  $\mu_B$  via  $f_t^{-1}$  and  $\mu_B \otimes \mathbb{K} = \mu_0$

2. The classification of  $Alg_2$  yields two isomorphic classes.

Let  $\{e_1, e_2\}$  be a basis of  $\mathbb{K}^2$ .

$$\mathcal{A}_1 : \mu_1(e_1, e_i) = \mu_1(e_i, e_1) = e_i \quad i = 1, 2; \quad \mu_1(e_2, e_2) = e_2.$$

$$\mathcal{A}_0 : \mu_0(e_1, e_i) = \mu_0(e_i, e_1) = e_i \quad i = 1, 2; \quad \mu_0(e_2, e_2) = 0.$$

Consider the formal deformation  $\mu_t$  of  $\mu_0$  defined by

$$\mu_t(e_1, e_i) = \mu_t(e_i, e_1) = e_i \quad i = 1, 2; \quad \mu_t(e_2, e_2) = t e_2.$$

Then  $\mu_t$  is isomorphic to  $\mu_1$  through the change of basis given by the matrix  $f_t = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$ ,  $\mu_t = f_t \cdot \mu_1$ .

We have  $\mu_0 = \lim_{t \rightarrow 0} f_t \cdot \mu_1$ , thus  $\mu_0$  is a degeneration of  $\mu_1$ . Since  $\mu_1$  is isomorphic to  $\mu_t$ , it is a deformation of  $\mu_0$ . We can obtain the same result over complex numbers if we consider the parameter  $t$  infinitesimal in  $\mathbb{C}^*$ .

#### 5.4.1 Graded algebra

In this Section, we give a relation between an algebra and its associated graded algebra.

**Theorem 5.2** *Let  $\mathcal{A}$  be an algebra over  $\mathbb{K}$  and  $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \dots \subseteq \mathcal{A}_n \subseteq \dots$  an algebra filtration of  $\mathcal{A}$ ,  $\mathcal{A} = \bigcup_{n \geq 0} \mathcal{A}_n$ . Then the graded algebra  $gr(\mathcal{A}) = \bigoplus_{n \geq 1} \mathcal{A}_n / \mathcal{A}_{n-1}$  is a formal degeneration of  $\mathcal{A}$ .*

**Proof 6** *Let  $\mathcal{A}[[t]]$  be a power series ring in one variable  $t \in \mathbb{K}$  over  $\mathcal{A}$ ,  $\mathcal{A}[[t]] = \mathcal{A} \otimes \mathbb{K}[[t]] = \bigoplus_{n \geq 0} \mathcal{A} \otimes t^n$ .*

*We denote by  $\mathcal{A}_t$  the Rees algebra associated to the filtered algebra  $\mathcal{A}$ ,  $\mathcal{A}_t = \bigoplus_{n \geq 0} \mathcal{A}_n \otimes t^n$ . The Rees algebra  $\mathcal{A}_t$  is contained in the algebra  $\mathcal{A}[[t]]$ .*

*For every  $\lambda \in \mathbb{K}$ , we set  $\mathcal{A}_{(\lambda)} = \mathcal{A}_t / ((t - \lambda) \cdot \mathcal{A}_t)$ . For  $\lambda = 0$ ,  $\mathcal{A}_{(0)} = \mathcal{A}_t / (t \cdot \mathcal{A}_t)$ . The algebra  $\mathcal{A}_{(0)}$  corresponds to the graded algebra  $gr(\mathcal{A})$  and  $\mathcal{A}_{(1)}$  is isomorphic to  $\mathcal{A}$ . In fact, we suppose that the parameter  $t$  commutes with the elements of  $\mathcal{A}$  then  $t \cdot \mathcal{A}_t = \bigoplus_{n \geq 0} \mathcal{A}_n \otimes t^{n+1}$ . It follows that  $\mathcal{A}_{(0)} = \mathcal{A}_t / (t \cdot \mathcal{A}_t) = (\bigoplus_n \mathcal{A}_n \otimes t^n) / (\bigoplus_n \mathcal{A}_n \otimes t^{n+1}) = \bigoplus_{n \geq 0} \mathcal{A}_n / \mathcal{A}_{n-1} = gr(\mathcal{A})$ . By using the linear map from  $\mathcal{A}_t$  to  $\mathcal{A}$  where the image of  $a_n \otimes t^n$  is  $a_n$ , we have  $\mathcal{A}_{(1)} = \mathcal{A}_t / ((t - 1) \cdot \mathcal{A}_t) \cong \mathcal{A}$ .*

*If  $\lambda \neq 0$ , the change of parameter  $t = \lambda T$  shows that  $\mathcal{A}_{(\lambda)}$  is isomorphic to  $\mathcal{A}_{(1)}$ . This ends the proof that  $\mathcal{A}_{(0)} = gr(\mathcal{A})$  is a degeneration of  $\mathcal{A}_{(1)} \cong \mathcal{A}$ .*

## 5.5 Connection between degeneration and deformation

The following proposition gives a connection between degeneration and deformation.

**Proposition 5.2** *Let  $\mathcal{A}_0$  and  $\mathcal{A}_1$  be two algebras in  $Alg_n$ . If  $\mathcal{A}_0$  is a degeneration of  $\mathcal{A}_1$  then  $\mathcal{A}_1$  is a deformation of  $\mathcal{A}_0$ .*

In fact, let  $\mathcal{A}_0 = \lim_{t \rightarrow 0} f_t \cdot \mathcal{A}_1$  be a formal degeneration of  $\mathcal{A}$  then  $\mathcal{A}_t = f_t \cdot \mathcal{A}_1$  is a formal deformation of  $\mathcal{A}_0$ .

In the contraction sense such a property follows directly from the definition.

In the global viewpoint, we get also that every degeneration can be realized by a global deformation. The base of the deformation is the completion of the discrete valuation  $\mathbb{K}$ -algebra (inductive limit of  $\mu_n = \mu_B \otimes B/m_B^{n+1}$ ).

**Remark 5.1** *The converse is, in general, false. The following example shows that there is no duality in general between deformation and degeneration.*

We consider, in  $Alg_4$ , the family  $\mathcal{A}_t = \mathbb{C}\{x, y\} / \langle x^2, y^2, yx - txy \rangle$  where  $\mathbb{C}\{x, y\}$  stands for the free associative algebra with unity and generated by  $x$  and  $y$ .

Two algebras  $\mathcal{A}_t$  and  $\mathcal{A}_s$  with  $t \cdot s \neq 1$  are not isomorphic. They are isomorphic when  $t \cdot s = 1$ .

Thus,  $\mathcal{A}_t$  is a deformation of  $\mathcal{A}_0$  but the family  $\mathcal{A}_t$  is not isomorphic to one algebra and cannot be written  $\mathcal{A}_t = f_t \cdot \mathcal{A}_1$ .

We have also more geometrically the following proposition :

**Proposition 5.3** *If an algebra  $\mathcal{A}_0$  is in the boundary of the orbit of  $\mathcal{A}_1$ , then this degeneration defines a nontrivial deformation of  $\mathcal{A}_0$ .*

With the contraction viewpoint, we can characterize the perturbation arising from degeneration by :

**Proposition 5.4** *Let  $\mu$  be a perturbation over hypercomplex numbers of an algebra with a multiplication  $\mu_1$ . Then  $\mu$  arises from a contraction if there exists an element  $\mu_0$  with structure constants in  $\mathbb{C}$  belonging to the orbit of  $\mu$ .*

In fact, if  $\mu$  is a perturbation of  $\mu_0$  and there exists an algebra with multiplication  $\mu_1$  such that:  $\mu \in \vartheta(\mu_1)$  and  $\mu \simeq \mu_0$ . Thus  $\mu_0$  is a contraction of  $\mu_1$ . This shows that the orbit of  $\mu$  passes through a point of  $Alg_n$  over  $\mathbb{C}$ .

## 5.6 Degenerations with $f_t = v + t \cdot w$

Let  $f_t = v + tw$  be a family of endomorphisms where  $v$  is a singular linear map and  $w$  is a regular linear map. The aim of this section is to find a necessary and sufficient conditions on  $v$  and  $w$  such that a degeneration of a given algebra  $\mathcal{A} = (V, \mu)$  exists. We can set  $w = id$  because  $f_t = v + tw = (v \circ w^{-1} + t) \circ w$  which is isomorphic to  $v \circ w^{-1} + t$ . Then with no loss of generality we can consider the family  $f_t = \varphi + t \cdot id$  from  $V$  into  $V$  where  $\varphi$  is a singular map. The vector space  $V$  can be decomposed by  $\varphi$  under the form  $V_R \oplus V_N$  where  $V_R$  and  $V_N$  are  $\varphi$ -invariant defined in a canonical way such that  $\varphi$  is surjective on  $V_R$  and nilpotent on  $V_N$ . Let  $q$  be the smallest integer such that  $\varphi^q(V_N) = 0$ . The inverse of  $f_t$  exists on  $V_R$  and is equal to  $\varphi^{-1}(t\varphi^{-1} + id)^{-1}$ . But on  $V_N$ , since  $\varphi^q = 0$ , it is given by

$$\frac{1}{\varphi + t \cdot id} = \frac{1}{t} \cdot \frac{1}{\varphi/t + id} = \frac{1}{t} \cdot \sum_{i=0}^{\infty} \left(-\frac{\varphi}{t}\right)^i = \frac{1}{t} \cdot \sum_{i=0}^{q-1} \left(-\frac{\varphi}{t}\right)^i$$

It follows that

$$f_t^{-1} = \begin{cases} \varphi^{-1} (t\varphi^{-1} + id)^{-1} & \text{on } V_R \\ \frac{1}{t} \cdot \sum_{i=0}^{q-1} \left(-\frac{\varphi}{t}\right)^i & \text{on } V_N \end{cases}$$

Let  $f_t = \varphi + t \cdot id$  be a family of linear maps on  $V$ , where  $\varphi$  is a singular map. The action of  $f_t$  on  $\mu$  is defined by  $f_t \cdot \mu = f_t^{-1} \circ \mu \circ f_t \times f_t$  then

$$\begin{aligned} f_t \cdot \mu(x, y) &= f_t^{-1} \circ \mu(f_t(x), f_t(y)) \\ &= f_t^{-1}(\mu(\varphi(x), \varphi(y)) + t(\mu(\varphi(x), y) + \mu(x, \varphi(y))) + t^2\mu(x, y)) \end{aligned}$$

Since every element  $v$  of  $V$  decomposes in  $v = v_R + v_N$ , we set

$$\begin{aligned} A &= \mu(x, y) = A_R + A_N, \\ B &= \mu(\varphi(x), y) + \mu(x, \varphi(y)) = B_R + B_N \\ C &= \mu(\varphi(x), \varphi(y)) = C_R + C_N \end{aligned}$$

Then

$$f_t \cdot \mu(x, y) = \varphi^{-1} (t\varphi^{-1} + id)^{-1} (t^2 A_R + t B_R + C_R) + \frac{1}{t} \cdot \sum_{i=0}^{q-1} \left(-\frac{\varphi}{t}\right)^i (t^2 A_N + t B_N + C_N)$$

If the parameter  $t$  goes to 0, then  $\varphi^{-1} (t\varphi^{-1} + id)^{-1} (t^2 A_R + t B_R + C_R)$  goes to  $\varphi^{-1} (C_R)$ . The limit of the second term is :

$$\begin{aligned} &\lim_{t \rightarrow 0} \frac{1}{t} \cdot \sum_{i=0}^{q-1} \left(-\frac{\varphi}{t}\right)^i (t^2 A_N + t B_N + C_N) \\ &= \lim_{t \rightarrow 0} \left( \frac{id}{t} - \frac{\varphi}{t^2} + \frac{\varphi^2}{t^3} - \cdots (-1)^{q-1} \frac{\varphi^{q-1}}{t^q} \right) (t^2 A_N + t B_N + C_N) \\ &= \lim_{t \rightarrow 0} t A_N - \varphi(A_N) + \left( \frac{id}{t} - \frac{\varphi}{t^2} + \frac{\varphi^2}{t^3} - \cdots (-1)^{q-1} \frac{\varphi^{q-3}}{t^{q-2}} \right) \varphi^2(A_N) + \\ &\quad B_N - \left( \frac{id}{t} - \frac{\varphi}{t^2} + \frac{\varphi^2}{t^3} - \cdots (-1)^{q-1} \frac{\varphi^{q-2}}{t^{q-1}} \right) \varphi(B_N) + \\ &\quad \left( \frac{id}{t} - \frac{\varphi}{t^2} + \frac{\varphi^2}{t^3} - \cdots (-1)^{q-1} \frac{\varphi^{q-1}}{t^q} \right) C_N \\ &= \lim_{t \rightarrow 0} t A_N + B_N - \varphi(A_N) + \\ &\quad \left( \frac{id}{t} - \frac{\varphi}{t^2} + \frac{\varphi^2}{t^3} - \cdots (-1)^{q-1} \frac{\varphi^{q-1}}{t^q} \right) (\varphi^2(A_N) - \varphi(B_N) + C_N) \end{aligned}$$

This limit exists if and only if

$$\varphi^2(A_N) - \varphi(B_N) + C_N = 0$$

which is equivalent to

$$\varphi^2(\mu(x, y)_N) - \varphi(\mu(\varphi(x), y)_N) - \varphi(\mu(x, \varphi(y))_N) + \mu(\varphi(x), \varphi(y))_N = 0$$

And the limit is  $B_N - \varphi(A_N)$ .

**Proposition 5.5** *The degeneration of an algebra with a multiplication  $\mu$  exists if and only if the condition*

$$\varphi^2 \circ \mu_N - \varphi \circ \mu_N \circ (\varphi \times id) - \varphi \circ \mu_N \circ (id \times \varphi) + \mu_N \circ (\varphi \times \varphi) = 0$$

where  $\mu_N(x, y) = (\mu(x, y))_N$ , holds. And it is defined by

$$\mu_0 = \varphi^{-1} \circ \mu_R \circ (\varphi \times \varphi) + \mu_N \circ (\varphi \times id) + \mu_N \circ (id \times \varphi) - \varphi \circ \mu_N$$

**Remark 5.2** *Using a more algebraic definition of a degeneration, Fialowski and O'Halloran [8] show that we have also a notion of universal degeneration and such degeneration exists for every finite dimensional Lie algebra. The definition and the result holds naturally in the associative algebra case.*

## 6 Rigidity

An algebra which has no nontrivial deformations is called rigid. In the finite dimensional case, this notion is related geometrically to open orbits in  $Alg_n$ . The Zariski closure of open orbit determines an irreducible component of  $Alg_n$ .

**Definition 11** *An algebra  $\mathcal{A}$  is called algebraically rigid if  $H^2(\mathcal{A}, \mathcal{A}) = 0$ .*

*An algebra  $\mathcal{A}$  is called formally rigid if every formal deformation of  $\mathcal{A}$  is trivial.*

*An algebra  $\mathcal{A}$  is called geometrically rigid if its orbit is Zariski open.*

We have the following equivalence due to Gerstenhaber and Schack [14].

**Proposition 6.1** *Let  $\mathcal{A}$  be a finite-dimensional algebra over a field of characteristic 0. Then formal rigidity of  $\mathcal{A}$  is equivalent to geometric rigidity.*

As seen in theorem (2.1),  $H^2(\mathcal{A}, \mathcal{A}) = 0$  implies that every formal deformation is trivial, then algebraic rigidity implies formal rigidity. But the converse is false. An example of algebra which is formally rigid but not algebraically rigid,  $H^2(\mathcal{A}, \mathcal{A}) \neq 0$ , was given by Gerstenhaber and Schack in positive characteristic and high dimension [14]. We do not know such examples in characteristic 0. However, there are many rigid Lie algebras, in characteristic 0, with a nontrivial second Chevalley-Eilenberg cohomology group.

Now, we give a sufficient condition for the formal rigidity of an algebra  $\mathcal{A}$  using the following map

$$Sq : \begin{aligned} H^2(\mathcal{A}, \mathcal{A}) &\rightarrow H^3(\mathcal{A}, \mathcal{A}) \\ \mu_1 &\mapsto Sq(\mu_1) = \mu_1 \circ \mu_1 \end{aligned}$$

Let  $\mu_1 \in Z^2(\mathcal{A}, \mathcal{A})$ ,  $\mu_1$  is integrable if  $Sq(\mu_1) = 0$ . If we suppose that  $Sq$  is injective then  $Sq(\mu_1) = 0$  implies that the cohomology class  $\mu_1 = 0$ . Then every integrable infinitesimal is equivalent to the trivial cohomology class. Therefore every formal deformation is trivial.

**Proposition 6.2** *If the map  $Sq$  is injective then  $\mathcal{A}$  is formally rigid.*

We have the following definition for rigid complex algebra with the perturbation viewpoint.

**Definition 12** *A complex algebra of  $Alg_n$  with multiplication  $\mu_0$  is infinitesimally rigid if every perturbation  $\mu$  of  $\mu_0$  belongs to the orbit  $\vartheta(\mu_0)$ .*

This definition characterizes the open sets over hypercomplex field (with metric topology). Since open orbit (with metric topology) is Zariski open [5]. Then

**Proposition 6.3** *For finite dimensional complex algebras, infinitesimal rigidity is equivalent to geometric rigidity, thus to formal rigidity.*

In the global viewpoint we set the following two concepts of rigidity.

**Definition 13** *Let  $\mathbb{B}$  be a commutative algebra over a field  $\mathbb{K}$  and  $m$  be a maximal ideal of  $\mathbb{B}$ . An algebra  $\mathcal{A}$  is called  $(\mathbb{B}, m)$ -rigid if every global deformation parametrized by  $(\mathbb{B}, m)$  is isomorphic to  $\mathcal{A}$  (in the push-out sense).*

*An algebra is called globally rigid if for every commutative algebra  $\mathbb{B}$  and a maximal ideal  $m$  of  $\mathbb{B}$  it is  $(\mathbb{B}, m)$ -rigid*

The global rigidity implies the formal rigidity but the converse is false. Fi-  
allowksi and Schlichenmaier show that over complex field the Witt algebra which is  
algebraically and formally rigid is not globally rigid. They use families of Krichever-  
Novikov type algebras [11].

Finally, we summarize the link between the different concepts of rigidity in the  
following theorem.

**Theorem 6.1** *Let  $\mathcal{A}$  be a finite dimensional algebra over a field  $\mathbb{K}$  of characteristic  
zero. Then*

$$\begin{aligned} \text{algebraic rigidity} &\implies \text{formal rigidity} \\ \text{formal rigidity} &\iff \text{geometric rigidity} \iff (\mathbb{K}[[t]], t\mathbb{K}[[t]])\text{-rigidity} \\ \text{global rigidity} &\implies \text{formal rigidity} \end{aligned}$$

In particular, if  $\mathbb{K} = \mathbb{C}$

$$\text{infinitesimal rigidity} \iff \text{formal rigidity}$$

Recall that semisimple algebras are algebraically rigid. They are classified by  
Wedderburn's theorem. The classification of low dimensional rigid algebras is known  
until  $n < 7$ , see [12],[31] and [28]. The classification of 6-dimensional rigid algebras  
was given using perturbation methods in [28].

## 7 The algebraic varieties $Alg_n$

A point in  $Alg_n$  is defined by  $n^3$  parameters, which are the structure constants  $C_{ij}^k$ ,  
satisfying a finite system of quadratic relations given by the associativity condition.  
The orbits are in 1-1-correspondence with the isomorphism classes of  $n$ -dimensional  
algebras. The stabilizer subgroup of  $\mathcal{A}$  ( $stab(\mathcal{A}) = \{f \in GL_n(\mathbb{K}) : \mathcal{A} = f \cdot \mathcal{A}\}$ ) is  
 $Aut(\mathcal{A})$ , the automorphism group of  $\mathcal{A}$ . The orbit  $\vartheta(\mathcal{A})$  is identified with the  
homogeneous space  $GL_n(\mathbb{K}) / Aut(\mathcal{A})$ . Then

$$\dim \vartheta(\mathcal{A}) = n^2 - \dim Aut(\mathcal{A})$$

The orbit  $\vartheta(\mathcal{A})$  is provided, when  $\mathbb{K} = \mathbb{C}$  (a complex field), with the structure of  
a differentiable manifold. In fact,  $\vartheta(\mathcal{A})$  is the image through the action of the Lie  
group  $GL_n(\mathbb{K})$  of the point  $\mathcal{A}$ , considered as a point of  $Hom(V \otimes V, V)$ . The Zariski  
tangent space to  $Alg_n$  at the point  $\mathcal{A}$  corresponds to  $Z^2(\mathcal{A}, \mathcal{A})$  and the tangent space  
to the orbit corresponds to  $B^2(\mathcal{A}, \mathcal{A})$ .

The first approach to the study of varieties  $Alg_n$  is to establish classifications of  
the algebras up to isomorphisms for a fixed dimension. Some incomplete classifications  
were known by mathematicians of the last centuries : R.S. Peirce (1870), E.  
Study (1890), G. Voghera (1908) and B.G. Scorza (1938). In 1974 [12], P. Gabriel  
has defined, the *scheme*  $Alg_n$  and gave the classification, up to isomorphisms, for  
 $n \leq 4$  and G. Mazzola, in 1979 [31], has studied the case  $n = 5$ . The number of  
different isomorphic classes grows up very quickly, for example there are 19 classes  
in  $Alg_4$  and 59 classes in  $Alg_5$ .

The second approach is to describe the irreducible components of a given alge-  
braic variety  $Alg_n$ . This problem has already been proposed by Study and solved

by Gabriel for  $n \leq 4$  and Mazzola for  $n = 5$ . They used mainly the formal deformations and degenerations. The rigid algebras have a special interest, an open orbit of a given algebra is dense in the irreducible component in which it lies. Then, its Zariski closure determines an irreducible component of  $Alg_n$ , i.e. all algebras in this irreducible component are degenerations of the rigid algebra and there is no algebra which degenerates to the rigid algebra. Two nonisomorphic rigid algebras correspond to different irreducible components. So the number of rigid algebra classes gives a lower bound of the number of irreducible components of  $Alg_n$ . Note that not all irreducible components are Zariski closure of open orbits.

Geometrically,  $\mathcal{A}_0$  is a degeneration of  $\mathcal{A}_1$  means that  $\mathcal{A}_0$  and  $\mathcal{A}_1$  belong to the same irreducible component in  $Alg_n$ .

The following statement gives an invariant which is stable under perturbations [28], it was used to classify the 6-dimensional complex rigid associative algebras and induces an algorithm to compute the irreducible components.

**Theorem 7.1** *Let  $\mathcal{A}$  be a  $\mathbb{C}$ -algebra in  $Alg_n$  with a multiplication  $\mu_0$  and  $X_0$  be an idempotent of  $\mu_0$ . Then, for every perturbation  $\mu$  of  $\mu_0$  there exists an idempotent  $X$  such that  $X \simeq X_0$ .*

This has the following consequence : The number of idempotents linearly independant does not decrease by perturbation.

Then, we deduce a procedure that finds the irreducible components by solving some algebraic equations [29], this procedure finds all multiplication  $\mu$  in  $Alg_n$  with  $p$  idempotents, then the perturbations of each such  $\mu$  are studied. From this it can be decided whether  $\mu$  belongs to a new irreducible component. By letting  $p$  run from  $n$  down to 1, one finds all irreducible components. A computer implementation enables to do the calculations.

Let us give the known results in low dimensions :

dimension	2	3	4	5	6
irreducible components	1	2	5	10	> 21
rigid algebras	1	2	4	9	21

The asymptotic number of parameters in the system defining the algebraic variety  $Alg_n$  is  $\frac{4n^3}{27} + O(n^{8/3})$ , see [35]. Any change of basis can reduce this number by at most  $n^2$  ( $= \dim(GL(n, \mathbb{K}))$ ). The number of parameters for  $Alg_2$  is 2 and for  $Alg_3$  is 6. For large  $n$  the number of irreducible components  $alg_n$  satisfies  $\exp(n) < alg_n < \exp(n^4)$ , see [31].

Another way to study the irreducible components is the notion of compatible deformations introduced by Gerstenhaber and Giaquinto [16]. Two deformations  $\mathcal{A}_t$  and  $\mathcal{A}_s$  of the same algebra  $\mathcal{A}$  are compatible if they can be joined by a continuous family of algebras. When  $\mathcal{A}$  has finite dimension  $n$ , this means that  $\mathcal{A}_t$  and  $\mathcal{A}_s$  lie on a common irreducible component of  $Alg_n$ . They proved the following theorem

**Theorem 7.2** *Let  $\mathcal{A}$  be a finite dimensional algebra with multiplication  $\mu_0$  and let  $\mathcal{A}_t$  and  $\mathcal{A}_s$  be two deformations of  $\mathcal{A}$  with multiplications  $\alpha_t = \mu_0 + tF_1 + t^2F_2 + \dots$  and  $\beta_s = \mu_0 + sH_1 + s^2H_2 + \dots$*

*If  $\mathcal{A}_t$  and  $\mathcal{A}_s$  are compatible then the classes  $f$  and  $h$  of the cocycles  $F$  and  $H$  satisfies  $[f, g]_G = 0$ , i.e.  $[F, H]_G$  must be a coboundary.*

The theorem gives a necessary condition for the compatibility of the deformations. It can be used to show that two deformations of  $\mathcal{A}$  lie on different irreducible components of  $Alg_n$ .

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